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New Approach to Guidance Law Design

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The proposed new guidance law design approach enables us to create laws that improve the effectiveness of the proportional navigation law. This new class of the guidance laws is developed based on the Lyapunov method. The new laws will augment the proportional navigation law which is a part of these laws; additional components are selected based on the negative definiteness of the derivative of the Lyapunov function. The guidance law components are determined based on the partial stability of the guidance system dynamics under consideration and only with respect to the line-of-sight derivative. The comparative analysis of the modified proportional navigation algorithms with the traditional proportional navigation law showed that the new guidance laws guarantee shorter homing time requirements and larger capture area.

Nomenclature

$\boldsymbol{a}_M$ = missile acceleration
$\boldsymbol{a}_T$ = target acceleration
$\boldsymbol{r}$ = target-to-missile range
t = time
u = guidance law
$\boldsymbol{V}_M$ = missile velocity
V_T = target velocity
$\boldsymbol{y}$ = relative separation between a missile and target perpendicular to the *OX* reference axis
λ = line-of-sight angle

I. Introduction

PROPORTIONAL navigation (PN) has attracted a considerable amount of interest in the literature related to missile guidance and continues to be a benchmark for new missile guidance laws. The detailed analytical study of this empirical guidance law for nonmaneuvering and maneuvering targets was undertaken in Refs. 1–3. Capture regions and conditions for the existence of capture regions were also obtained in Ref. 4.

The basic philosophy behind PN guidance is that missile acceleration should nullify the line-of-sight (LOS) rate. Analysis of PN guidance for the homing stage was usually undertaken for nonmaneuvering targets assuming a constant closing velocity.[1,2] The so-called augmented PN law and other modifications of the PN law were obtained based mostly on the relationships established for nonmaneuvering targets.[1,2]

The results from the theory of linear multivariable control systems applied to homing guidance (where linear approximation can be justified) enable one both to evaluate the performance of the guidance system and to generate modified PN laws.[1,2,5] Guidance laws utilizing the idea of PN and based on the results of control theory related to sliding modes and systems with variable structure (see, e.g., Ref. 6) cannot be considered practical for missile guidance applications. The practical realization of systems with sliding modes is limited because of chatter, and related simplified control laws need rigorous justification and testing. A systematic framework for an almost sliding mode control that eliminates chatter was given in Ref. 7. However, this approach was not used in Ref. 6 and other applications of sliding mode controls in the guidance systems. Also, in the presence of a maneuvering target the sliding mode area depends on the target acceleration, and for small LOS derivatives the sliding mode can disappear. A variable structure (different from the ones considered, e.g., in Ref. 6) that requires measurement of target acceleration is needed.

The empirical PN law was obtained also as a solution of an optimization problem (see, e.g., Ref. 1), which justifies this law as an optimal one corresponding to a certain quadratic performance index. The game approach to guidance laws based on the theory of differential games with a quadratic performance index was considered in Ref. 8. The guidance laws developed counteract target maneuvers better than the ordinary PN law.

However, any optimal guidance law assumes that the trajectory of a maneuvering target as well as the time to go and/or the intercept point is known. In practice, such information is unknown and can only be evaluated approximately. The accuracy of prediction influences significantly the accuracy of the intercept.[9]

Taking into account that the PN law is a widely accepted guidance law and has been tested in practice, it is of interest to consider the possibility of its improvement. The approach offered in this paper can also be considered another justification of the PN law. Moreover, this approach enables us to offer other laws that will improve the effectiveness of the PN law for maneuvering and nonmaneuvering targets. A new class of the PN guidance laws developed is obtained as the solution of a stability problem using the Lyapunov method. The Lyapunov function is chosen as a square of the LOS derivative. The applicability of the laws is determined by the negative definiteness of the derivative of the Lyapunov function. The module of the Lyapunov function derivative along the trajectory of the engagement system can be used as a performance index of guidance laws. It is important to mention that the guidance laws are determined based on the partial stability of the system dynamics under consideration only with respect to the LOS derivative.[10,11]

II. Problem Formulation and Guidance Correction Controls

PN is the guidance law that implements parallel navigation, which is defined by the rule $\dot{\lambda}(t) = 0$ with an additional requirement $\dot{r}(t) < 0$, where $\lambda(t)$ is the LOS angle with respect to a reference axis and $\boldsymbol{r}(t)$ represents the target-to-missile range.[2]

To describe the missile–target engagement dynamics, we consider planar engagements and use a Cartesian frame of coordinates (see Fig. 1) with the origin O of an inertial reference coordinate system: $\boldsymbol{y}(t)$ is the relative separation between the missile and target perpendicular to the OX reference axis; $\boldsymbol{V}_M$, $\boldsymbol{a}_M$, V_T, and $\boldsymbol{a}_T$ are the missile and target velocity and acceleration, respectively.

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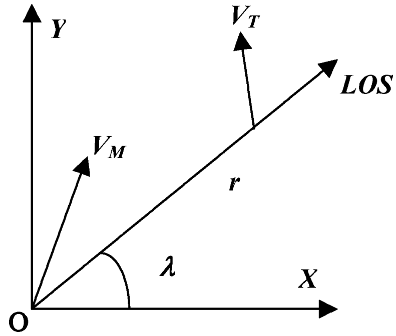

Fig. 1 Engagement geometry.

Using a small-angle approximation, the expressions for the LOS angle and its derivatives can be presented in the following form (see Appendix A):

$$\lambda(t) = \boldsymbol{y}(t)/\boldsymbol{r}(t) \tag{1}$$

$$\ddot{\lambda} = a_1(t)\lambda(t) - a_2(t)\dot{\lambda}(t) + b(t)\ddot{y}(t) \tag{2}$$

where

$$a_1(t) = \ddot{r}(t)/\boldsymbol{r}(t) \tag{3}$$

$$a_2(t) = 2\dot{r}(t)/\boldsymbol{r}(t) \tag{4}$$

$$b(t) = 1/\boldsymbol{r}(t) \tag{5}$$

$$\ddot{y}(t) = -\boldsymbol{a}_M(t) + \boldsymbol{a}_T(t) \tag{6}$$

Let $x_1 = \lambda(t)$ and $x_2 = \dot{\lambda}(t)$. The missile–target engagement is described by the following system of first-order differential equations:

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -a_1(t)x_1 - a_2(t)x_2 - b(t)u + b(t)f \tag{7}$$

where the control $u = \boldsymbol{a}_M(t)$ and disturbance $f = \boldsymbol{a}_T(t)$.

The guidance problem can be formulated as the problem of choosing control u to guarantee the asymptotic stability of system (7) with respect to x_2. (Because in reality we deal with a finite problem, for simplicity and a more rigorous utilization of the term "asymptotic stability" we assume disturbance to be a vanishing function; i.e., it contains a factor $e^{-\varepsilon t}$, where ε is an infinitely small positive number.)

First, let us consider the case of a nonaccelerating target (i.e., $f = 0$), the assumption that accompanies the main relations used in PN guidance. For the Lyapunov function

$$Q = \tfrac{1}{2}cx_2^2 \tag{8}$$

where c is a positive coefficient, its derivative along any trajectory of Eqs. (7) equals

$$\dot{Q} = cx_2[-a_1(t)x_1 - a_2(t)x_2 - b(t)u] \tag{9}$$

Under the near-collision course assumption,[2] $\ddot{r}(t) = 0$ (i.e., $a_1(t) = 0$), and the control law that guarantees the negative definiteness of Eq. (9), that is, the asymptotic stability of Eqs. (7) with respect to x_2, can be presented in the form (see Appendix B) $u = kx_2$, $kb(t) + a_2(t) > 0$ or

$$k > -a_2(t)/b(t) \tag{10}$$

Introducing the closing velocity $v_{\text{cl}} = -\dot{r}(t)$ and the effective navigation ratio N, expression (10) can be written as $k > 2v_{\text{cl}}$ and the control law can be presented as

$$u = Nv_{\text{cl}}\dot{\lambda}, \qquad N > 2 \tag{11}$$

which is the well-known property established for the PN guidance law.[1,2] PN guidance law (11) is called admissible if it guarantees intercept for a finite time T. We consider the PN class of guidance laws that have a form such as Eq. (11) or contain Eq. (11) as a component. Despite the fact that even the PN laws of the form of Eq. (11) with various N were compared by experiments, we will introduce a criterion of comparison that has a certain physical justification. Because PN is the guidance law that implements parallel navigation $[\dot{\lambda}(t) = 0]$, we will compare the laws belonging to the PN class by their closeness to parallel navigation.

Of course, the most reliable performance index should evaluate the guidance law during the entire engagement time. However, this time is unknown and itself depends on the guidance law implemented. To avoid this "catch 22" situation, we assume that the guidance law with $\dot{\lambda}(t)$ tending to zero faster (closer to parallel navigation) at each t is preferable.

The module of the Lyapunov function derivative $|\dot{Q}(t)|$ [see Eq. (9)] will be the performance index for comparing the PN laws and creating the new ones. Proceeding in this way, we change the finite interval engagement problem for a specific infinite interval partial stability problem. The Lyapunov approach will be used to compare and design controls: guidance laws.

Let us assume that there exists a capture range domain over which the control (guidance law) $u(t)$ guarantees engagement $[x_2(t) \to 0]$. Then it can be concluded (see Appendix B) that the guidance law

$$u = Nv_{\text{cl}}\dot{\lambda}(t) + N_1\dot{\lambda}^3(t), \qquad N > 2, \qquad N_1 > 0 \tag{12}$$

is better than PN law (11).

It was shown that PN law (11) can be improved by using a complex exponential-type function of time $N(t)$ instead of a constant N.[8] This function depends on the predicted time to go. Its calculation presents certain difficulties for utilization of such laws in practice. Guidance law (12) can be written in the form of Eq. (11) as

$$u(t) = \left[N + (N_1/v_{\text{cl}})\dot{\lambda}^2(t)\right]v_{\text{cl}}\dot{\lambda}(t) = N(t)v_{\text{cl}}\dot{\lambda}(t) \tag{13}$$

with a time-varying coefficient $N(t)$, which formally is an exponential-type function. [Asymptotic with respect to x_2 solution of Eqs. (7), under the assumption mentioned earlier, is an exponential type function.] The form of Eq. (13) looks similar to the guidance laws considered in Ref. 8. In contrast to the law with variable $N(t)$,[8] guidance law (12) does not require special complex computations.

Let us consider the case of a maneuvering target but assuming a small $\ddot{r}(t)$ so that we can neglect $a_1(t)$. From the condition of negative definiteness of the derivative of Lyapunov function (8),

$$\dot{Q} = cx_2[-a_1(t)x_1 - a_2(t)x_2 - b(t)u + b(t)f] \tag{14}$$

we can derive the guidance laws (see Appendix B)

$$u = Nv_{\text{cl}}\dot{\lambda} + \boldsymbol{a}_T(t), \qquad N > 2 \tag{15}$$

$$u = Nv_{\text{cl}}\dot{\lambda}(t) + N_1\dot{\lambda}^3(t) + \boldsymbol{a}_T(t), \qquad N > 2, \qquad N_1 > 0 \tag{16}$$

where the acceleration term is $0.5N$ times less than in the augmented PN (APN) law obtained for step maneuvers.[1,2] Law (15) is given as a possible law to compare with the existing augmented law. Later a more precise expression will be given. It is easy to establish the negative definiteness of Eq. (14) for $a_1(t) \neq 0$ and $a_2(t) \leq 0$ if the control u is

$$u = Nv_{\text{cl}}\dot{\lambda}(t) + N_1\dot{\lambda}^3(t) - N_2\lambda(t)\ddot{r}(t) + N_3\boldsymbol{a}_T(t)$$

$$N > 2, \qquad N_1 > 0$$

$$N_2 \begin{matrix}\geq 1\\ \leq 1\end{matrix} \quad \text{if} \quad \text{sign}[\ddot{r}(t)\dot{\lambda}(t)\lambda(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix}$$

$$N_3 \begin{matrix}\leq 1\\ \geq 1\end{matrix} \quad \text{if} \quad \text{sign}[\boldsymbol{a}_T(t)\dot{\lambda}(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix} \tag{17}$$

III. General Case

Instead of using small linear approximation (1), we will consider a general nonlinear case. The expressions for the LOS angle and its derivatives can be presented in the following form:

$$\sin[\lambda(t)] = \boldsymbol{y}(t)/\boldsymbol{r}(t) \tag{18}$$

$$\ddot{\lambda}(t)\cos[\lambda(t)] - \dot{\lambda}^2(t)\sin[\lambda(t)] = -a_1(t)\lambda(t) - a_2(t)\cos[\lambda(t)]\dot{\lambda}(t) + b_1\ddot{y}(t) \tag{19}$$

and instead of system (7) we have the nonlinear system

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_2^2\tan x_1 - a_1(t)(x_1/\cos x_1) - a_2(t)x_2 - b(t)u + b(t)f \tag{20}$$

It is important to mention that in considering a linear approximation of the trigonometric functions in Eq. (20) we will not obtain linear system (7). There will be the additional nonlinear term $x_2^2x_1$ (for small x_1, $x_2^2\tan x_1 \approx x_2^2x_1$). The linearization for small LOS angles at this stage is more rigorous than linearization (1) and sequential differentiation (2) as it was done in many publications. The main reason for using Eqs. (1) and (2) is the difficulty in dealing with nonlinear differential equations.

The derivative of Lyapunov function (8) along any trajectory of Eq. (20) is

$$\dot{Q} = cx_2\left[x_2^2\tan x_1 - a_1(t)(x_1/\cos x_1) - a_2(t)x_2 - b(t)u + b(t)f\right]$$

or

$$\dot{Q} = c\left[x_2^3\tan x_1 - a_1(t)(x_1x_2/\cos x_1) - a_2(t)x_2^2 - b(t)x_2u + b(t)x_2f\right] \tag{21}$$

The negative definiteness of Eq. (21) can be guaranteed by the control guidance law

$$u = Nv_{\text{cl}}\dot{\lambda}(t) + N_1\dot{\lambda}^3(t) - N_2\lambda(t)\{\ddot{r}(t)/\cos[\lambda(t)]\} - N_0\boldsymbol{r}(t)\dot{\lambda}^2(t)\tan[\lambda(t)] + N_3\boldsymbol{a}_T(t), \qquad N > 2,\ N_1 > 0 \tag{22}$$

$$N_0 \begin{matrix}\geq 1\\ \leq 1\end{matrix} \quad \text{if} \quad \text{sign}\{\dot{\lambda}(t)\tan[\lambda(t)]\} \begin{matrix}\leq 0\\ \geq 0\end{matrix}$$

$$N_2 \begin{matrix}\geq 1\\ \leq 1\end{matrix} \quad \text{if} \quad \text{sign}[\ddot{r}(t)\dot{\lambda}(t)\lambda(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix}$$

$$N_3 \begin{matrix}\leq 1\\ \geq 1\end{matrix} \quad \text{if} \quad \text{sign}[\boldsymbol{a}_T(t)\dot{\lambda}(t)] \begin{matrix}\leq 0\\ \geq 0\end{matrix}$$

Guidance law (22) can be presented as the sum of the main PN law and additional correcting controls:

$$u = Nv_{\text{cl}}\dot{\lambda} + \sum_{k=0}^{3} u_k \tag{23}$$

where

$$u_0 = -N_0\boldsymbol{r}(t)\dot{\lambda}^2(t)\tan[\lambda(t)] \tag{24}$$

$$u_1 = N_1\dot{\lambda}^3(t) \tag{25}$$

$$u_2 = -N_2\lambda(t)\{\ddot{r}(t)/\cos[\lambda(t)]\} \tag{26}$$

$$u_3 = N_3\boldsymbol{a}_T(t) \tag{27}$$

For small LOS angles and short homing ranges (the case that was discussed mostly in the guidance literature), the term $x_2^3\tan x_1$ in Eq. (21) is smaller than a dominant x_2^2 component. That is why the analysis of linear system (7) is justified if such conditions are satisfied. For a larger spectrum of LOS angles the u_0 component is needed. The effectiveness of the u_1 correction was discussed earlier. The u_2 correction is needed for maneuvering targets and when the second derivative of range is not small enough. APN term (27) differs from the well-known one [see also Eq. (15)] that was obtained rigorously for step maneuvers but was recommended to be used for all types of maneuvers.[1,2] The sign $[\boldsymbol{a}_T(t)\dot{\lambda}(t)]$ factor reflects the dependence of the correction on the target behavior. Each of the controls u_k ($k = 0, 1, 2, 3$) increases the effectiveness of the PN law with respect to the criterion chosen. The number of the controls applied in practice should depend on the problem under consideration (target distances, LOS angles, maneuvering or nonmaneuvering targets, etc., as well as the system's ability to realize the correction control in practice).

Taking into account that we consider a class of the modified PN laws, control corrections (24–27) are considered the means of improving PN law (11), extending its area of applicability. The coefficients N_0–N_3 (constant or time-varying) can be determined based on simulation results of the whole missile system taking into account the autopilot limits on missile acceleration, airframe dynamics, and other factors (i.e., the same way as the classical PN values $N = 3$–4 were established). An approach to their selection is given in the next section. The effectiveness of the preceding approach is demonstrated in the following example.

IV. Illustrative Example

Here we consider a realistic example of a tail-controlled aerodynamic missile operating at high altitude to illustrate the effectiveness of the new guidance law and compare it to PN guidance results.[12] The flight-control dynamics are assumed to be presented by a third-order transfer function

$$W(s) = \frac{1 - s^2/\omega_z^2}{(1+\tau s)[1 + (2\zeta/\omega)s + s^2/\omega^2]}$$

with damping ζ and natural frequency ω similar to that in Refs. 12 ($\zeta = 0.7$ and $\omega = 20$ rad/s), the flight control system time constant $\tau = 0.5$ s, and the right half-plane zero, $\omega_z = 5$ rad/s.

As mentioned in Ref. 12, at high altitudes, where the airframe zero frequency ω_z can be low, optimal guidance, similar to that in Ref. 8 for the single-lag model, has no advantage when compared to PN and can produce even worse results. Miss distance, when using optimal guidance, increases as the airframe zero decreases. A new optimal guidance law that accounts for the presence of airframe zero was developed and tested in Ref. 12. It works better than a PN law but cannot be presented as a closed-form solution and is developed numerically and stored as a tabulated function of time depending on several factors.

The performance of the new guidance laws will be compared to PN. We assume that the PN gain $N = 4$ and the closing velocity $V_{\text{cl}} = 1219.2$ m/s and consider the homing stage when the LOS angle is relatively small so that the expressions of Sec. II can be utilized. As in Ref. 18, two error sources are considered: a $3g$ constant target maneuver and $1mr$ of range-independent angle measurement noise. The missile acceleration limit is $10g$.

A simplified model of the missile engagement is presented in Fig. 2.[1,12] Here R_{TM} is the range $\boldsymbol{r}$ between a missile and a target and $\hat{R}_{\text{TM}}$ is its estimate. The measurement of the LOS angle λ_k^* is corrupted by noise. A pseudomeasurement of relative position, y_k^*, is created by a multiplication of λ_k^* by $\hat{R}_{\text{TM}}$. The Kalman filter then provides optimal estimates of relative position, relative velocity, and a target acceleration.[1,12] Three guidance laws are considered: PN; nonlinear guidance, discussed in the preceding sections, without measurements of target acceleration; and nonlinear guidance utilizing measurements of target acceleration. The nonlinear guidance law has the following form:

$$u = 4v_{\text{cl}}\dot{\lambda}(t) + N_1\dot{\lambda}^3(t) + N_3\boldsymbol{a}_T(t) \tag{28}$$

[For the linearized engagement model in Fig. 2 we assume a constant closing velocity so that the term with the second derivative $\ddot{r}(t)$ in

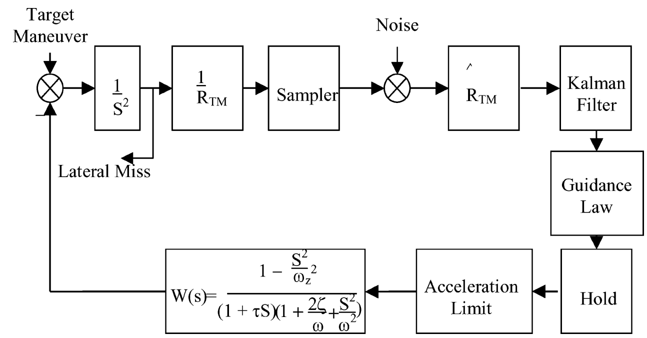

Fig. 2 Missile guidance model.

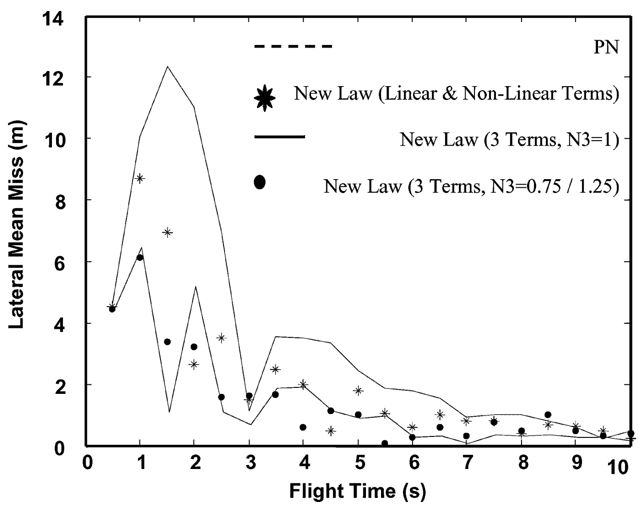

Fig. 3 New guidance laws yield improved performance.

Eq. (17) equals zero.] We used the discrete form of Eq. (28) and the estimates of $\dot{\lambda}$ in the form[1] (symbol ^ denotes estimates)

$$\hat{\dot{\lambda}}_k = \frac{\hat{y}_k + \hat{\dot{y}}_k t_{\text{go}}}{v_{\text{cl}} t_{\text{go}}^2}$$

The results of a Monte Carlo simulation, analogous to Ref. 1, for a step target maneuver and zero initial conditions are presented in Fig. 3. The mean absolute value of the resultant lateral miss is given based on 50 simulation trials.

The nonlinear term with gain, $N_1 = 40{,}000\ v_{\text{cl}}$, significantly improves the performance of a missile when compared to the PN guidance law. A further improvement is reached by measuring target acceleration: by using a constant gain $N_3 = 1$ (solid curve) or a time-dependent gain $N_3 = 0.75$, we obtain 1.25 [data denoted by dots; see also Eq. (17)]. Each component of Eq. (28) increases $|\dot{Q}(t)|$. This fact enables us to choose the gains N_i ($i = 0, \ldots, 3$) sequentially.

The PN law reacts almost identically on various changes of LOS rate (assuming that the closing velocity does not vary drastically); that is, small and fast changes of LOS result in proportional changes of acceleration. According to Eqs. (14) and (16), by increasing N in the PN law, we can decrease the LOS rate faster. But this will increase the level of noise when the LOS rate becomes small and, hence, the accuracy of guidance is decreased. Moreover, big gains can make the whole guidance system is unstable. From a purely physical consideration we can assume that the system with a variable gain that is larger when the LOS rate is large and smaller when LOS rate is small will act better than the traditional PN system. The second component of Eq. (28) [see Eq. (13)] with a properly chosen N_1 serves this purpose.

In the aforementioned example, N_1 was chosen based on the value of the LOS angle rate estimates for the case of the PN guidance law [i.e., when only the first component of Eq. (28) was used], which was about 0.006 rad/s at the beginning of the homing stage and significantly less (three times and more) at the end of the homing stage. For the given N_1 value we indirectly increased N in the PN law at the beginning of the homing stage [according to Eq. (13), almost 30%]. However, the "cubic term" has a negligible influence at the end of the homing stage [see Eq. (13)].

V. Conclusions

Based on the Lyapunov method a new approach to missile guidance law design was developed. It was shown that a class of guidance laws that implements parallel navigation is wider than the PN law and the additional terms can significantly improve the performance of the PN guidance law. The new guidance laws utilize the same parameters as the PN and APN laws and can be realized in practice.

Appendix A: LOS Derivatives

The expression for the second derivative of the LOS angle follows from the two consecutive differentiations of $\lambda(t) = \boldsymbol{y}(t)/\boldsymbol{r}(t)$:

$$\dot{\lambda}(t) = \frac{\dot{y}(t)\boldsymbol{r}(t) - \boldsymbol{y}(t)\dot{r}(t)}{r^2(t)} = \frac{\dot{y}(t)}{\boldsymbol{r}(t)} - \frac{\lambda(t)\dot{r}(t)}{\boldsymbol{r}(t)} \tag{A1}$$

$$\begin{aligned}\ddot{\lambda}(t) &= \frac{\ddot{y}(t)\boldsymbol{r}(t) - \dot{y}(t)\dot{r}(t)}{r^2(t)} - \frac{[\dot{\lambda}(t)\dot{r}(t) + \lambda(t)\ddot{r}(t)]\boldsymbol{r}(t) - \lambda(t)\dot{r}^2(t)}{r^2(t)} \\ &= \frac{\ddot{y}(t) - \dot{\lambda}(t)\dot{r}(t) - \lambda(t)\ddot{r}(t)}{\boldsymbol{r}(t)} - \frac{\dot{r}(t)}{\boldsymbol{r}(t)}\frac{[\dot{y}(t) - \lambda(t)\dot{r}(t)]}{\boldsymbol{r}(t)} \\ &= \frac{\ddot{y}(t) - \dot{\lambda}(t)\dot{r}(t) - \lambda(t)\ddot{r}(t) - \dot{\lambda}(t)\dot{r}(t)}{\boldsymbol{r}(t)} \\ &= \frac{\ddot{y}(t) - 2\dot{\lambda}(t)\dot{r}(t) - \lambda(t)\ddot{r}(t)}{\boldsymbol{r}(t)}\end{aligned} \tag{A2}$$

By introducing time-varying coefficients (3–5), expression (A2) can be presented in the form of Eq. (2).

Appendix B: Lyapunov Approach to Control Law Design

The Lyapunov approach to control law design can be explained in the following way (more rigorous formulations and theorems can be found, e.g., in Ref. 11): if there exist positive definite functions $Q(x, t)$ and $R(x, t)$ so that the derivative $\dot{Q}$ with respect to t along any trajectory of the system of equations that describes the control system under consideration ($\boldsymbol{x}$ and u are its state vector and control, respectively) satisfy the inequality

$$\dot{Q} = \dot{Q}(x, u, t) \leq -R(x, t) \tag{B1}$$

then the system is stabilized by control u, which can be determined from this inequality.

To apply this sufficient condition in practice, the preceding indicated positive definite forms must be found. Unfortunately, there are no universal recommendations on how to find these forms. The relation between $Q(x, t)$ and $R(x, t)$ was established for the so-called linear quadratic optimal control problems (Riccati-type equations).[11] Based on this, the design procedure was expanded on a certain class of nonlinear systems.[11]

However, for special types of equations it is not difficult to find $Q(x, t)$ and $R(x, t)$ satisfying inequality (B1). Next, the control law design procedure based on the Lyapunov approach is demonstrated for the guidance problem [see Eqs. (7) and (8); for simplicity, we consider in Eq. (7) $f = 0$]. By choosing $Q(x, t)$ in the form of Eq. (8) and $R(x, t) = c_1 x_2^2$, where c_1 is a positive coefficient, inequality (B1) can be written as [see also Eq. (9)]

$$\dot{Q} = c x_2 [-a_1(t)x_1 - a_2(t)x_2 - b(t)u] \leq -c_1 x_2^2 \tag{B2}$$

or

$$[-a_2(t) + c_1/c]x_2^2 - a_1(t)x_1x_2 - b(t)x_2u \leq 0 \tag{B3}$$

It follows from Eq. (B3) that for $a_1(t) = 0$ and $cl \ll c$ the control $u = kx_2$ [see Eq. (11)] stabilizes system (7) if k satisfies Eq. (10). For $R(x, t) = c_1 x_2^2 + c_2 x_2^4$, where c_2 is a positive coefficient, instead of Eq. (B2) we have

$$\dot{Q} = cx_2\left[-a_1(t)x_1 - a_2(t)x_2 - b(t)u\right] \le -c_1 x_2^2 - c_2 x_2^4 \quad \text{(B4)}$$

or

$$\left[-a_2(t) + c_1/c\right]x_2^2 - a_1(t)x_1x_2 + (c_2/c)x_2^4 - b(t)x_2u \le 0 \quad \text{(B5)}$$

It is easy to conclude that for $a_1(t) = 0$ and $c_1 \ll c$ and the control $u = kx_2 + N_1 x_2^3$, where k satisfies Eq. (10) and $N_1 > 0$, the left-hand side of Eq. (B5) is negative definite, so that this control [see Eq. (12)] stabilizes system (7) with respect to x_2.

By including the additional term in R we imposed "harder" requirements on the rate of decreasing Q. Despite the fact that $|\dot{Q}|$ is used as a system estimate in some applications of the Lyapunov method (see, e.g., Ref. 11) it cannot be applied as a reliable criterion of quality of control systems. It serves only as an instantaneous criterion. The quality estimate of a control system includes (directly or indirectly) time of control. For example, an oscillatory long transient even with small amplitude in many cases is unacceptable. However, when choosing the guidance laws implementing parallel navigation, the only requirement is to be closer, as soon as possible, to zero LOS rate. The $|\dot{Q}|$ criterion reflects this requirement. This is why we can conclude that guidance law (12) is better than PN law (11).

If in Eq. (7) $f \ne 0$, instead of Eq. (B4) we have

$$\dot{Q} = cx_2\left[-a_1(t)x_1 - a_2(t)x_2 - b(t)u + b(t)f\right] \le -c_1 x_2^2 - c_2 x_2^4 \quad \text{(B6)}$$

An additional component in control $f = \boldsymbol{a}_T(t)$ "compensates" in Eq. (B6) for the $b(t)f$ term so that the control $u = kx_2 + N_1 x_2^3 + \boldsymbol{a}_T(t)$ [see Eq. (16)] stabilizes system (7) with respect to x_2. The Lyapunov approach is demonstrated here in detail for system (7) when $a_1(t) = 0$. In the analogous way, Eqs. (17) and (24–27) can be obtained.

References

[1]Zarchan, P. (ed.), *Tactical and Strategic Missile Guidance*, Vol. 176, Progress in Astronautics and Aeronautics, AIAA, Reston, VA, 1997.

[2]Shneydor, N. A., *Missile Guidance and Pursuit*, Horwood Publishing, Chichester, England, U.K., 1998.

[3]Guelman, M., "A Qualitative Study of Proportional Navigation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 7, No. 4, 1971, pp. 637–643.

[4]Ghose, D., "True Proportional Navigation with Maneuvering Target," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 30, No. 1, 1994, pp. 229–237.

[5]Gurfil, P., Jodorkovsky, M., and Guelman, M., "Neoclassical Guidance for Homing Missiles," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 3, 2001, pp. 452–459.

[6]Moon, J., Kim, K., and Kim, J., "Design of Missile Guidance Law via Variable Structure Control," *Journal of Guidance, Control, and Dynamics*, Vol. 24, No. 4, 2001, pp. 659–664.

[7]Yanushevsky, R. T., "An Approach to Design on Control Systems with Parametric-Coordinate Feedback," *IEEE Transactions on Automatic Control*, Vol. 36, No. 11, 1991, pp. 1293–1295.

[8]Ben-Asher, J. Z., and Yaesh, I., *Advances in Missile Guidance Theory*, edited by P. Zarchan, Vol. 180, Progress in Astronautics and Aeronautics, AIAA, Reston, VA, 1998.

[9]Kim, K. B., Kim, M. J., and Kwon, W. H., "Receding Horizon Guidance Laws with No Information on the Time-to-Go," *Journal of Guidance, Control, and Dynamics*, Vol. 23, No. 2, 2000, pp. 193–199.

[10]Rumyantsev, V. V., "On the Stability with Respect to a Part of the Variables," *Atti Convergno di Mecanica Razionale Symposia Mathematica, Monograf*, Yugoslavia, 1971.

[11]Yanushevsky, R. T., "A Controller Design for a Class of Nonlinear Systems Using the Lyapunov-Bellman Approach," *Journal of Dynamic Systems, Measurement and Control*, Vol. 114, No. 9, 1992, pp. 390–394.

[12]Zarchan, P., Greenberg, E., and Alpert, J., "Improving the High Altitude Performance of Tail-Controlled Endoatmospheric Missiles," AIAA Paper 2002-4770, Aug. 2002.